Three-level models solvable in terms of the Clausen function

2003 J. Phys. A: Math. Gen. 36 7331

(http://iopscience.iop.org/0305-4470/36/26/308)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 7331–7335 PII: S0305-4470(03)61451-4

Three-level models solvable in terms of the Clausen function

Artur Ishkhanyan[1] **and Kalle-Antti Suominen**[2,3]

[1] Engineering Center of Armenian National Academy of Sciences, Ashtarak-2, 378410, Armenia
[2] Helsinki Institute of Physics, PL 64, FIN-00014 Helsingin yliopisto, Finland
[3] Department of Physics, University of Turku, FIN-20014 Turun yliopisto, Finland

E-mail: artur@ec.sci.am

Received 26 March 2003, in final form 13 May 2003
Published 18 June 2003
Online at stacks.iop.org/JPhysA/36/7331

Abstract

The problem of analytical integrability of the three-level problem by reduction of the time-dependent Schrödinger equations to the third-order linear differential equation satisfied by the generalized hypergeometric functions ${}_3F_2$ is considered. A total of 12 infinite classes of models solvable in terms of these functions is found, most of which are new and others are generalizations of the previously known families.

PACS numbers: 02.30.Gp, 02.30.Hq

1. Introduction

Analytic solutions of the time-dependent Schrödinger equations for three-state quantum systems are extensively applied in the investigation of many physical problems in the theory of interaction of radiation with matter during the past decade. With the help of these solutions, a number of term crossing and pulse-shape effects, population transfer problems, diffraction and interference effects, and other aspects of non-adiabatic transitions in quantum systems were investigated (see, e.g., [1–7]). For this reason a considerable amount of research has been devoted to the exact solutions of the Schrödinger equations in terms of special functions [8–10]. The general method used for finding new integrable cases is the reduction of the Schrödinger equations to a standard differential equation with known solutions. A starting point for the present study is the observation that most non-trivial solutions to the three-level problem are expressed in terms of generalized hypergeometric functions. Since the three-level problem can be formulated in the form of a third-order ordinary differential equation, more attention should be thus paid to the solutions which are expressed in terms of hypergeometric functions ${}_1F_2$, ${}_2F_2$ and ${}_3F_2$ (only these hypergeometric functions obey third-order equations, see [11]). We herein concentrate on the solutions which may be expressed in terms of function ${}_3F_2$. There exist several classes of three-state models for which the Clausen function is used to construct the solution (see, e.g., [1, 2, 5, 6, 9, 10]). We generalize all the presently known solutions in terms of the Clausen function and derive a number of new classes of solvable models. Altogether we list here 12 infinite classes of integrable cases of the three-level problem. We hope that the presented solutions will be useful in studying different problems of quantum optics and atomic physics.

2. The integrable classes

In the rotating wave approximation, the time-dependent Schrödinger equations written for the population amplitudes $a_{1,2,3}$ of a three-level quantum system interacting with two quasi-resonant classical electromagnetic waves read

$$\mathrm{i}a_{1t} = U\,\mathrm{e}^{-\mathrm{i}\delta_1}a_2 \qquad \mathrm{i}a_{2t} = U\,\mathrm{e}^{+\mathrm{i}\delta_1}a_1 + V\,\mathrm{e}^{-\mathrm{i}\delta_2}a_3 \qquad \mathrm{i}a_{3t} = V\,\mathrm{e}^{+\mathrm{i}\delta_2}a_2 \tag{1}$$

where $U(t)$ and $V(t)$ are the Rabi frequencies and $\delta_{1,2}(t)$ present the detuning modulation functions. By eliminating a_2 and a_3 from this system, one may write the problem in the form of a third-order ordinary differential equation for the first-level amplitude a_1:

$$\begin{aligned} a_{1ttt} &+ \left(2\mathrm{i}\delta_{1t} - 2\frac{U_t}{U} + \mathrm{i}\delta_{2t} - \frac{V_t}{V}\right)a_{1tt} + \left[\left(\mathrm{i}\delta_{1t} - \frac{U_t}{U}\right)_t + U^2 + V^2 + \left(\mathrm{i}\delta_{1t} - \frac{U_t}{U}\right)\right. \\ &\left.\times\left(\mathrm{i}\delta_{1t} - \frac{U_t}{U} + \mathrm{i}\delta_{2t} - \frac{V_t}{V}\right)\right]a_{1t} + U^2\left(\mathrm{i}\delta_{1t} + \frac{U_t}{U} + \mathrm{i}\delta_{2t} - \frac{V_t}{V}\right)a_1 = 0. \end{aligned} \tag{2}$$

Now, our aim is to find real functions $U(t)$, $V(t) > 0$ and $\delta_{1,2}(t)$, for which the transformation of the dependent and independent variables

$$u = \varphi(z)a_1(z) \qquad z = z(t) \tag{3}$$

reduces equation (2) to the equation satisfied by the Clausen function

$$u_{zzz} + \frac{f_1 z + f_0}{z(1-z)}u_{zz} + \frac{g_1 z + g_0}{z^2(1-z)}u_z + \frac{h_1 z + h_0}{z^3(1-z)}u = 0. \tag{4}$$

As is known, the general solution of this equation is given in terms of five-parameter generalized hypergeometric functions ${}_3F_2$ [11]

$$u = \sum_{m=1}^{3} C_m z^{\eta_m}{}_3F_2\big(\eta_m - \xi_1, \eta_m - \xi_2, \eta_m - \xi_3; 1 + \eta_m - \eta_{m_1}, 1 + \eta_m - \eta_{m_2}; z\big) \tag{5}$$

where $C_{1,2,3}$ are arbitrary constants, the parameters η_1, η_2, η_3 and ξ_1, ξ_2, ξ_3 are determined from the following cubic characteristic equations

$$\eta(\eta-1)(\eta-2) + f_0\eta(\eta-1) + g_0\eta + h_0 = 0 \tag{6}$$

$$\xi(\xi-1)(\xi-2) - f_1\xi(\xi-1) - g_1\xi - h_1 = 0 \tag{7}$$

and $\left(\eta_{m_1}, \eta_{m_2}\right)$ is the complementary to η_m pair from the triad (η_1, η_2, η_3).

According to the general approach based on the class property of the solvable cases [10], one should rewrite equation (2) substituting throughout variable t by z and equate the coefficients of the obtained equation and equation (4), the latter being previously transformed through the transformation of the independent variable (3). Further, if functions $U^*(z)$, $V^*(t)$ and $\delta^*_{1,2}(z)$ (which are referred to as basic models) are the solutions of the obtained system of equations then functions $U(t)$, $V(t)$ and $\delta_{1,2}(t)$ defined by the relations

$$U(t) = U^*(z)\frac{\mathrm{d}z}{\mathrm{d}t} \qquad V(t) = V^*(z)\frac{\mathrm{d}z}{\mathrm{d}t} \qquad \frac{\mathrm{d}\delta_{1,2}(t)}{\mathrm{d}t} = \delta^*_{1,2z}(z)\frac{\mathrm{d}z}{\mathrm{d}t} \tag{8}$$

form a class of functions for which the solution of the initial three-level problem is expressed in terms of generalized hypergeometric functions by formula (5). Here $z = z(t)$ is any physically reasonable complex-valued function of real variable t (time) performing one-to-one mapping $t \longleftrightarrow z$.

Following the previous work [10], we search for the basic solutions of the form

$$U^* = U_0^* z^{k_1}(1-z)^{n_1} \qquad V^* = V_0^* z^{k_2}(1-z)^{n_2} \qquad \delta^*_{1,2z} = \frac{\beta_{1,2}}{z} + \frac{\gamma_{1,2}}{1-z} \tag{9}$$

where the parameters $k_{1,2}$ and $n_{1,2}$ are integers or half-integers.

If we confine ourselves to the simplest case $\varphi(z) \equiv 1$, then the outlined procedure will lead us to the following equations:

$$\frac{2\mathrm{i}\beta_1 + \mathrm{i}\beta_2 - 2k_1 - k_2}{z} + \frac{2\mathrm{i}\gamma_1 + \mathrm{i}\gamma_2 + 2n_1 + n_2}{1-z} = \frac{f_0 + f_1 z}{z(1-z)} \tag{10}$$

$$\left(-\frac{\mathrm{i}\beta_1 - k_1}{z^2} + \frac{\mathrm{i}\gamma_1 + n_1}{(1-z)^2}\right) + U_0^{*2} z^{2k_1}(1-z)^{2n_1} + V_0^{*2} z^{2k_2}(1-z)^{2n_2} + \left(\frac{\mathrm{i}\beta_1 - k_1}{z} + \frac{\mathrm{i}\gamma_1 + n_1}{1-z}\right) \times \left(\frac{\mathrm{i}\beta_1 + \mathrm{i}\beta_2 - k_1 - k_2}{z} + \frac{\mathrm{i}\gamma_1 + \mathrm{i}\gamma_2 + n_1 + n_2}{1-z}\right) = \frac{g_0 + g_1 z}{z^2(1-z)} \tag{11}$$

$$U_0^{*2} z^{2k_1}(1-z)^{2n_1}\left(\frac{\mathrm{i}\beta_1 + \mathrm{i}\beta_2 + k_1 - k_2}{z} + \frac{\mathrm{i}\gamma_1 + \mathrm{i}\gamma_2 - n_1 + n_2}{1-z}\right) = \frac{h_0 + h_1 z}{z^3(1-z)}. \tag{12}$$

From the first equation of this system we have

$$f_0 = 2\mathrm{i}\beta_1 + \mathrm{i}\beta_2 - 2k_1 - k_2 \tag{13}$$

$$f_1 + f_0 = 2\mathrm{i}\gamma_1 + \mathrm{i}\gamma_2 + 2n_1 + n_2. \tag{14}$$

The other two equations give

$$g_0 = (\mathrm{i}\beta_1 - k_1)(\mathrm{i}\beta_1 + \mathrm{i}\beta_2 - k_1 - k_2 - 1) + [(U^{*2} + V^{*2})z^2]_{z=0} \tag{15}$$

$$g_1 + g_0 = (\mathrm{i}\beta_1 - k_1)(\mathrm{i}\gamma_1 + \mathrm{i}\gamma_2 + n_1 + n_2) + (\mathrm{i}\gamma_1 + n_1)(\mathrm{i}\beta_1 + \mathrm{i}\beta_2 - k_1 - k_2) + [(U^{*2} + V^{*2})(1-z)]_{z=1} \tag{16}$$

and

$$h_0 = (U^{*2}z^2)_{z=0}(\mathrm{i}\beta_1 + \mathrm{i}\beta_2 + k_1 - k_2) \tag{17}$$

$$h_1 + h_0 = (U^{*2})_{z=1}(\mathrm{i}\gamma_1 + \mathrm{i}\gamma_2 - n_1 + n_2) + [U^{*2}(1-z)]_{z=1}(\mathrm{i}\beta_1 + \mathrm{i}\beta_2 + k_1 - k_2). \tag{18}$$

Besides, the structure of equations (15)–(18) suggests imposing several additional restrictions on the involved parameters. First, one can easily show that should be $k_{1,2}, n_{1,2} \geqslant -1$ and $k_{1,2} + n_{1,2} < 0$. Further, it is deduced that $n_1 \neq -1$ and $k_1 \neq 0$. Finally, it is necessary to eliminate the term proportional to $1/(1-z)^2$ in the second equation of (11). This leads to the following relations

$$[(U^{*2} + V^{*2})(1-z)^2]_{z=1} + (\mathrm{i}\gamma_1 + n_1)(\mathrm{i}\gamma_1 + \mathrm{i}\gamma_2 + n_1 + n_2 + 1) = 0 \tag{19}$$

$$U^{*2} + V^{*2} = \frac{A}{z^2} + \frac{B}{z(1-z)} + \frac{C}{(1-z)^2} \tag{20}$$

Table 1. Twelve basic models integrable in terms of the Clausen function.

N	k_1	n_1	k_2	n_2	U^*	V^*	$\delta^*_{1z}, \delta^*_{2z}$	Restriction
1	-1	$-1/2$	-1	$-1/2$	$\frac{U_0^*}{z\sqrt{1-z}}$	$\frac{V_0^*}{z\sqrt{1-z}}$	$\frac{\beta_{1,2}}{z} \pm \frac{\gamma_1}{1-z}$	$\gamma_1 + \gamma_2 = 0$
2	-1	$-1/2$	$-1/2$	$-1/2$	$\frac{U_0^*}{z\sqrt{1-z}}$	$\frac{V_0^*}{\sqrt{z(1-z)}}$	$\frac{\beta_{1,2}}{z} \pm \frac{\gamma_1}{1-z}$	$\gamma_1 + \gamma_2 = 0$
3	-1	0	-1	-1	$\frac{U_0^*}{z}$	$\frac{\sqrt{\gamma_1(\gamma_1+\gamma_2)}}{z(1-z)}$	$\frac{\beta_{1,2}}{z} + \frac{\gamma_{1,2}}{1-z}$	$V_0^{*2} = \gamma_1(\gamma_1 + \gamma_2)$
4	-1	0	$-1/2$	-1	$\frac{U_0^*}{z}$	$\frac{\sqrt{\gamma_1(\gamma_1+\gamma_2)}}{\sqrt{z}(1-z)}$	$\frac{\beta_{1,2}}{z} + \frac{\gamma_{1,2}}{1-z}$	$V_0^{*2} = \gamma_1(\gamma_1 + \gamma_2)$
5	-1	0	0	-1	$\frac{U_0^*}{z}$	$\frac{\sqrt{\gamma_1(\gamma_1+\gamma_2)}}{1-z}$	$\frac{\beta_{1,2}}{z} + \frac{\gamma_{1,2}}{1-z}$	$V_0^{*2} = \gamma_1(\gamma_1 + \gamma_2)$
6	-1	0	-1	$-1/2$	$\frac{U_0^*}{z}$	$\frac{V_0^*}{z\sqrt{1-z}}$	$\frac{\beta_1}{z}, \frac{\beta_2}{z} + \frac{\gamma_2}{1-z}$	$\gamma_1 = 0$
7	-1	0	-1	0	$\frac{U_0^*}{z}$	$\frac{V_0^*}{z}$	$\frac{\beta_1}{z}, \frac{\beta_2}{z} + \frac{\gamma_2}{1-z}$	$\gamma_1 = 0$
8	-1	0	$-1/2$	$-1/2$	$\frac{U_0^*}{z}$	$\frac{V_0^*}{\sqrt{z(1-z)}}$	$\frac{\beta_1}{z}, \frac{\beta_2}{z} + \frac{\gamma_2}{1-z}$	$\gamma_1 = 0$
9	$-1/2$	$-1/2$	-1	$-1/2$	$\frac{U_0^*}{\sqrt{z(1-z)}}$	$\frac{V_0^*}{z\sqrt{1-z}}$	$\frac{\beta_{1,2}}{z} \pm \frac{\gamma_1}{1-z}$	$\gamma_1 + \gamma_2 = 0$
10	$-1/2$	$-1/2$	$-1/2$	$-1/2$	$\frac{U_0^*}{\sqrt{z(1-z)}}$	$\frac{V_0^*}{\sqrt{z(1-z)}}$	$\frac{\beta_{1,2}}{z} \pm \frac{\gamma_1}{1-z}$	$\gamma_1 + \gamma_2 = 0$
11	$-1/2$	0	0	$-1/2$	$\frac{U_0^*}{\sqrt{z}}$	$\frac{U_0^*}{\sqrt{1-z}}$	$\frac{\beta_1}{z}, \frac{\beta_2}{z} + \frac{\beta_1+\beta_2}{1-z}$	$V_0^* = U_0^*, \gamma_1 = 0, \gamma_2 = \beta_1 + \beta_2$
12	$-1/2$	0	-1	$+1/2$	$\frac{U_0^*}{\sqrt{z}}$	$\frac{U_0^*\sqrt{1-z}}{z}$	$\frac{\beta_1}{z}, \frac{\beta_2}{z} + \frac{\beta_1+\beta_2}{1-z}$	$V_0^* = U_0^*, \gamma_1 = 0, \gamma_2 = \beta_1 + \beta_2$

where A, B and C are arbitrary constants. The examination of all the possible alternatives, taking into account the mentioned restrictions, leads us, finally, to 12 independent basic solutions. These functions, together with the necessary restrictions imposed on the involved parameters, are presented in table 1. To get a notion of the typical representatives of corresponding classes, one may apply a standard transformation $z = [1 + \tanh(t)]/2$.

The listed classes include all the presently known three-level models integrable in terms of generalized hypergeometric functions ${}_3F_2$. The tenth class, a class involving pulses of the same shape, is the class considered in detail by Carroll and Hioe [2]. The sixth and ninth classes are generalizations to weaker restrictions of two families introduced in [10]. The eleventh class was recently presented by us in [6]. Other classes are derived for the first time. As is seen from the table, these families present different symmetric and asymmetric pulse-shapes and detuning modulation functions that could be useful for investigation of pulse-shape and level crossing effects simultaneously. For instance, the fifth class presenting pulse shapes analogous to that considered by Laine–Stenholm [7] additionally describes delayed level crossings.

3. Summary

Thus, we have carried out a search for the cases of the three-level problem for which the solutions are written in terms of Clausen's generalized hypergeometric functions ${}_3F_2$. We have previously presented a ten-parametric class of models permitting general solution of the Schrödinger equations in the form of convergent power series [10]. Here we specify 12 infinite subclasses of that general class which are integrable in terms of the Clausen function. Some of these classes are generalizations of previously known families, while the rest of them are new. The obtained models may be applied for investigation of specific physical problems of quantum and atomic optics. For instance, the fifth class may be employed for study of multiple term-crossings (between different pairs of the levels) in three-state systems. Finally, we would like to note that the above analysis can be easily applied to other equations, e.g., to those that are obeyed by generalized hypergeometric functions ${}_1F_2$ and ${}_2F_2$.

Acknowledgments

The authors acknowledge support from the Academy of Finland (Project 50314). AI thanks the Helsinki Institute of Physics, where this research was accomplished, for kind hospitality, Nikolay Vitanov and Ashot Manukyan for useful discussions, as well as acknowledges the support by the International Science and Technology Center (grant A215), US Civilian Research and Development Foundation (grant NFSAT PH 100-02/12042), and Armenian National Science and Education Fund (grant PS-13).

References

[1] Carroll C E and Hioe F T 1990 *Phys. Rev.* A **42** 1522
Carroll C E and Hioe F T 1988 *J. Opt. Soc. Am.* B **5** 1335
[2] Carroll C E and Hioe F T 1987 *Phys. Rev.* A **36** 724
Carroll C E and Hioe F T 1988 *J. Math. Phys.* **29** 487
[3] Ishkhanyan A M and Melikdzanov D Yu 1988 *Dokl. Akad. Nauk Armenii* **86** 71
[4] Carroll C E and Hioe F T 1992 *Phys. Rev. Lett.* **68** 3523
[5] Ishkhanyan A M 2000 *Phys. Rev.* A **61** 063609
[6] Ishkhanyan A M and Suominen K-A 2002 *Phys. Rev.* A **65** 051403(R)
[7] Laine T A and Stenholm S 1996 *Phys. Rev.* A **53** 2501
[8] Carroll C E and Hioe F T 1989 *J. Phys. B: At. Mol. Opt. Phys.* **22** 2633
[9] Ishkhanyan A M 1994 *J. Contemp. Phys. (Armenian Natl Acad. Sci.)* **29** 9
[10] Ishkhanyan A M 2000 *J. Phys. A: Math. Gen.* **33** 5041
[11] Slater L J 1966 *Generalized Hypergeometric Functions* (Cambridge: Cambridge University Press) chapter 2, p 42